A NOTE ON RANDOMLY REGULAR GRAPHS

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Abstract: The graph $G$ is said to be a randomly $H$ graph if and only if any subgraph of $G$ without isolated vertices, which is isomorphic to a subgraph of $H$, can be extended to a subgraph $F$ of $G$ such that $F$ is isomorphic to $H$. In this paper the problem about randomly $H$ graphs is discussed, where $H$ is a connected regular graph.

Key words: Randomly $H$ graph, regular graph.

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INTRODUCTION

In 1951 Ore [13] studied arbitrarily traceable graphs which were later referred to as randomly eulerian graphs. This concept was later extended by Chartrand and White [5] and Erickson [8]. Chartrand and Kronk [2] in 1968 introduced the concept of randomly hamitonian graphs. There were also characterized these graphs. The analogical questions have been studied in [4], [6], [7], [14].

In 1986 Chartrand, Oellerman and Ruiz [3] generalized these concepts and introduced the term randomly $H$ graph as follows: Let $G$ be a graph containing a subgraph $H$ without isolated
vertices. Then G is called a randomly H graph if whenever F is a subgraph of G without isolated vertices that is isomorphic to a subgraph of H, then F can be extended to a subgraph H1 of G such that H1 is isomorphic to H. Thus, every nonempty graph is randomly K2 graph while every graph G without isolated vertices is a randomly G graph. The graph K4,4 in Figure 1 is not randomly C6, since the subgraph F1 of K4,4 cannot be extended to a subgraph of K4,4 isomorphic to C6.

The graph K3,3 is randomly H for every subgraph H of K3,3 (see [3]). In order to avoid a situation where only a complete graph would be randomly H, require in the definition of randomly H graph that H and F be without isolated vertices (see also [3]). In [11] a characterization of randomly Km,n graphs is given (H-closed graph is there used for the term randomly H graph). In [1], [10], [12] randomly complete n-partite graphs were studied and characterized.

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In [15] the problem of characterization of randomly Hr graphs (H-closed graph is there used for the term randomly H graph) on p vertices was given, where Hr is a connected p-vertex r regular graphs. The characterization of randomly Hr graphs in general seems to be difficult. But there exist several results for some special values of r, p, n.

In [15] the following theorem is proved:

THEOREM 1. ([15; Theorem 1]) Let G be n -vertex graph and Hr be connected p-vertex regular graph of degree r≥2, different from K3 and C4. Then G is randomly Hr graph if and only if G=Kn , for n >p.
In [14] the characterization of randomly \( H_{2p}^1 \) graphs (but in terms randomly matchable graphs) was given.

**THEOREM 2.** (see Sumner [15]). Graph \( G \) is randomly \( H_{2p}^1 \) graph if and only if \( G \) is \( K_{2p} \) or \( K_{p,p} \) or \( H_{2p}^1 \).

Chartrand, Oellerman and Ruiz in [3] the following theorem proved:

**THEOREM 3.** (see [3]). If \( H^2_p \) is a cycle on \( p \) vertices for \( p \geq 5 \), then randomly \( H^2_p \) graphs are \( K_n \) or \( H^2_p \) if \( p \) is odd and \( K^r_{p,p} \) or \( K^r_{p,p} \) if \( p \) is even \( (n \geq p) \).

In [15] also the following criterion for randomly graphs is given:

**THEOREM 4.** (see [15; Lemma 2]). Graph \( G \) is randomly \( H \) graph if and only if for every minimal system \( S = \{x_1, x_2, \ldots, x_k\} \) of boolean variables for which the boolean expression

\[
W = \prod_{H \subseteq G} \left( \sum_{e \in E(G) - E(H)} x_e \right)
\]

is true, \( F_s \not\subset H \), where \( F_s \) is the graph consisting of the edges \( \{e_1, e_2, \ldots, e_k\} \) corresponding to the boolean variables in \( S \).

This criterion works very good if \( G \) has not much more edges than \( H \).

Some results concerning to randomly \( H^r_p \) graphs on \( p \) vertices for \( r \geq 3 \) are given in this paper. We use the general notation and terminology of Harary [9]. It is followed from Theorems 2 and 3 that all known nontrivial randomly \( H^r_p \) graphs up to the present time are only bipartite graphs. Bipartite graphs have very significant part in study of randomly regular graphs. This fact is confirmed by Theorems 2 and 3 and it follows also from theorems bellow.

**THEOREM 5.** Let \( H^r_{2p} \) be a connected \( 2p \)-vertex \( r \) regular graph and \( K_{p,p} \subset H^r_{2p} \). Let \( G \) be a randomly \( H^r_{2p} \) graph, then \( G = H^r_{2p} \) or \( G = K_{2p} \).

**Proof.** If \( r = p \), then theorem follows from [11; Theorem 3]. Now, let \( r > p \), and \( G \) be a randomly \( H^r_{2p} \) graph. If \( G \neq H^r_{2p} \) then there exists a subgraph \( H \) of \( G \) which is isomorphic to \( H^r_{2p} \). Because of \( K_{p,p} \subset H^r_{2p} \), the vertex set of \( G \), \( V(G) = A \cup B \), where \( A = \{v_1, v_2, \ldots, v_p\} \), \( B = \{u_1, u_2, \ldots, u_p\} \) and \((v_i, u_j) \in E(G)\) for \( i = 1, 2, ..., p; j = 1, 2, ..., p \). From \( r > p \) it follows, that there exist edges between vertices of \( A \) (\( B \) respectively). Now, let \( e \in E(G) - E(H) \) and \( e = (u_i, u_j) \). Form a graph \( F \) as follows:
\[ \text{THEOREM 6.} \quad \text{Let } G = K_{p,p} - M_{p,p}, \text{ where } M_{p,p} \text{ be a matching on } 2p \text{ vertices. Then } K_{p,p} \text{ is randomly } G \text{ graph.} \]

Proof. We use the criterion from Theorem 4. The graph \( K_{p,p} \) contains exactly \( p! \) graphs \( H \) because of, there are \( p! \) matchings \( M_{p,p} \) on \( 2p \) vertices. Thus the boolean expression \( W \) has the following form:

\[
W = \prod_{P \in P} \left( x_{1,i_1} + x_{2,i_2} + \ldots + x_{p,i_p} \right),
\]

where the product \( \prod \) is done over all permutations of set \( \{1, 2, \ldots, p\} \). If the boolean expression is true, the \( F_S \) must contain at least one edge from every matching. Now, if \( F_S \subset H \), then there exists matching \( \{ e_{1,j}, e_{2,j}, \ldots, e_{p,j} \} = E( K_{p,p}) - E( H) \) such that \( F_s \cap \{ e_{1,j}, e_{2,j}, \ldots, e_{p,j} \} = \emptyset \). This is the contradiction to the fact that \( W \) is true.

From Theorems 2, 3 and 6 it follows that \( K_{p,p} \) is randomly \( H_{2p}^r \) graph, for \( r=1, 2 \) and \( p-1 \). For \( r \geq 3 \) and \( r \neq p-1 \) the graph \( K_{p,p} \) must not be randomly \( H_{2p}^r \) graph. For example, if \( p=6 \), the graph \( H \) in Figure 2 is 3-regular 12-vertex subgraph of \( K_{6,6} \), but the graph \( K_{6,6} \) is not the randomly \( H \) graph, since the subgraph \( F_1 \) of \( K_{6,6} \) that is isomorphic to a subgraph of \( H \) cannot be extended to a subgraph of \( K_{6,6} \) isomorphic to \( H \).

NOTE: Using \( (v, k, \lambda) \) designs, for every \( r \geq 3 \) there exists \( p \) such that we can construct \( r \)-regular \( 2p \)-vertex graph \( H \), for which \( K_{p,p} \) is not the randomly \( H \) graph.

COROLLARY. Graph \( K_{p,p} \) for \( p = 1, 2, 3, 4 \) is randomly \( H \) regular graph for every its connected regular subgraph \( H \).

Proof. It is easy to verify this assertion for \( K_{1,1}, K_{2,2} \) and \( K_{3,3} \). The graph \( K_{4,4} \) has a following connected regular subgraph: matching \( M_{4,4} \), cycle \( C \) and \( H = K_{4,4} - M_{4,4} \). The proof follows from Theorems 2, 3 and 6.
REFERENCES