INTEGRAL TREES AND PELL’S EQUATIONS

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A graph $G$ is called integral if all the zeros of the characteristic polynomial $P(G;x)$ are integers. In the present paper the class of integral trees of diameter 3 is studied. It is proved that the problem of characterizing integral trees of diameter 3 is equivalent with the problem of solving Pell’s Diophantine equations.

Key words: Tree, characteristic polynomial, integral tree, Pell’s equations.

1. Introduction

Let $G$ be a graph. The characteristic polynomial $P(G;x)$ of a graph $G$ is defined to be characteristic polynomial of the adjacency matrix of $G$. The spectrum of the adjacency matrix is also called the spectrum of $G$, and is denoted by $Sp(G)$. In general terminology we follow Harary [2]. In addition we use the terminology of Cvetkovic [1] and notation of [5], [6] and [7].

We say that $G$ has an integral spectrum if all the zeros of $P(G;x)$ are integers. A graph $G$ is called integral if it has an integral spectrum. The term of integral graph was introduced by Harary and Schwenk in [3]. In general, the problem of characterizing integral graphs seems to be difficult. However, if we restrict our attention to trees, the prospects are much better. The problem of integral trees was considered in [4], [5], [6], [8], [9].

In this paper we restrict our investigations to integral trees of diameter 3. Integral trees of diameter 3 were studied for example in [5], [8], [9]. The tree of diameter 3 is balanced, if all the vertices at the same distance from the centre are of the same degree and is denoted by $T(1;n_1)$. It is known (see [5], [8]) that balanced tree $T = T(1;n_1)$ is integral if and only if $n_1=s_1(s_1+1)$ where $s_1 \in \mathbb{N}$. For more details see [5]. The main concern of this paper is to give a relationship between integral trees of diameter 3 and Pell’s equations and construct integral trees of diameter 3 using them. For the notations and terminology of Pell’s equations see [10].

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The analogous balanced integral trees of diameter 6 characterization based on Pell’s equations is described in [4].

2. Preliminaries

We will use the theory of divisors and codivisors, which is useful in the spectral graph theory. More information about the theory of divisors can be found in [1], [5] and [7].

The most important property of divisor $D$ of a graph $G$ is that the characteristic polynomial $P(D;x)$ divides the characteristic polynomial of $G$ ([1; Theorem 4.5]), i.e. there exists a polynomial $P(C;x)$ such that

$$P(G;x) = P(D;x) \cdot P(C;x).$$

Note that in general a divisor $D$ of a graph $G$ is a directed graph with multiple edges and loops and its codivisor $C$ is a directed graph whose arcs are valued by plus or minus one.

Every tree of diameter 3 can be characterized by two parameters $a, b \in \mathbb{N}$ and denoted by $T(1; a \mid b)$. Its diagram is depicted in Fig. 1.

By considering a construction of divisor and codivisor of the tree $T(1; a \mid b)$ as given in [1; page 126] we get the following propositions.

**Proposition 1.** The characteristic polynomial of the corresponding divisor $D(1; a \mid b)$ of the tree $T(1; a \mid b)$ can be expressed by the formula

$$P(D(1; a \mid b)) = x^4 - (a + b + 1)x^2 + ab.$$

The diagram of the divisor is depicted in Fig. 2.

**Proposition 2.** The characteristic polynomial of a corresponding codivisor $C(1; a \mid b)$ of the tree $T(1; a \mid b)$ can be expressed by the formula

$$P(C(1; a \mid b)) = x^{a+b-2}.$$

The diagram of the codivisor is depicted in Fig. 3.

**Proposition 3.** The characteristic polynomial of the tree $T(1; a \mid b)$ can be expressed by the formula

$$P(T(1; a \mid b)) = (x^4 - (a + b + 1)x^2 + ab)x^{a+b-2}.$$

Hence the spectrum of codivisor $C(1; a \mid b)$ consists only of zeros. Now, it is sufficient to find positive integers $a, b$ for that the equation (1) has only integer zeros.
3. Results

If we want the tree $T(1; a \mid b)$ to be integral, equation (1) has to have only integer zeros. Because a spectrum of a tree is always symmetric, we have $S_p(D(1; a \mid b)) = \{r, \pm s\}$, where $r, s \in \mathbb{N}$. We have to find $a, b \in \mathbb{N}$ for which the equation (1) has integer zeros.

Holds
(4) $P(D(1; a \mid b)) = x^4 - (r^2 + s^2)x^2 + r^2s^2$.

By comparing equations (1) and (4) we have
(5) $a + b = r^2$,
(6) $a + b + 1 = r^2 + s^2$.

By elimination of $r$ we have
(7) $(a - s^2)(b - s^2) = s^2$.

Using substitutions
$$a - s^2 = s_1, \\
b - s^2 = s_2$$
in (6) and (7) it is easy to show by the calculus that
(8) $s_1s_2 = s^2$,
(9) $s_1 + s_2 + 1 = r^2 - s^2$,
(10) $(1 + s_1)(1 + s_2) = r^2$.

Now we have to find solutions $s_1, s_2 \in \mathbb{N}$ of equations (8) and (10).

It is easy to show that if $s_1 = s_2$, then $a = b = s_1(s_1 + 1)$, where $s_1 \in \mathbb{N}$ and $T(1; a \mid b)$ is balanced tree of diameter 3. (see [5])

Now let $s_1$ be arbitrary positive integer.

Using (8) we have $s_2 = \frac{s^2}{s_1}$.

Denote by $d$ the greatest common divisor of $s$ and $s_1$. Let $s = d \cdot s'$. Then $s^2 = d^2 \cdot (s')^2$.

Now let $s_1 = d \cdot s_1$. Since $d = (s, s_1)$ we have $(s', s_1) = 1$.

Since $s_2 = \frac{d^2 - s^2}{d \cdot s_1} = \frac{d \cdot s'^2}{s_1}$ and $(s', s_1) = 1$ we have $s_1 \mid d$ and $d = s_1 \cdot s_1''$.

Using previous formulas in (9) it is easy to show that
$$s_1 + s_2 + 1 = r^2 - s^2 = r^2 - d^2s'^2,$$
$$s_1 + 1 = r^2 - \frac{s_1d^2 + d}{s_1} s'^2$$

and
(11) $s_1 + 1 = r^2 - (d^2 + s_1'')s'^2$.

Notice that $s_1$ is the positive integer, $s_1'$ and $d$ are divisors of $s_1$ and $s_1 = d \cdot s_1'$.

Using
(12) $d = s_1' \cdot s_1''$

we have
(13) $s_1 = (s_1')^2 \cdot s_1''$.

That is why (11) is the Pell's equation with form
$$r^2 - D \cdot s^2 = L,$$

where
$$D = d^2 + s_1'' \\ \\
L = s_1 + 1.$$
If we want to find the solution of equations (8) and (10), we have to solve the Pell's equation (11).

**Theorem 1.** The tree \( T(1; a|b) \) is integral one and its spectrum is \( \pm r, \pm s, 0, \ldots, 0 \) if and only if the following formulas hold:

\[
\begin{align*}
a &= s^2 + s_1 \\
b &= s^2 + s_2 \\
s_2 &= \frac{s^2}{s_1} \\
s &= d \cdot s'_r
\end{align*}
\]

\((r, s')\) is the solution of the equation (11) where \( s_1 \) is arbitrary positive integer and both equations (12) and (13) holds for \( d, s'_1 \).

The proof of the necessary condition is above the Theorem 1. Conversely, using formulas from the Theorem 1 in (1) it is easy to show by the calculus that the tree \( T(1; a|b) \) is integral.

**Corollary 1.** Let \( s_1 = 1 \). It is easy to show using (12) and (13) that the equation (11) has form

\[
2 = r^2 - 2 s^2.
\]

All its solutions can be expressed by the formula \( r_n + s'_n \sqrt{2} = ( 2 + \sqrt{2} )(3 + 2 \sqrt{2} )^n \), where \( n = 0, 1, 2, 3, \ldots \) (see [9, Theorem 6.34]).

If \( n=0 \), then \((r, s) = (2, 1)\) and \( a = b = 2 \). We have found balanced integral tree \( T(1;2) \). (see [5])

If \( n=1 \), then \( r_1 + s'_1 \sqrt{2} = ( 2 + \sqrt{2} )(3 + 2 \sqrt{2} ) = ( 10 + 7 \sqrt{2} ) \). The ordered pair \((r,s') = (10,7)\) is the solution of the Pell's equation (14). Using formulas under the theorem we have \( a = 50, b = 98 \). We have found non-balanced integral tree \( T(1;50|98) \), which has the smallest number of vertices from all trees of this class. Its diagram is depicted in Fig. 4. The characteristic polynomial of its divisor is

\[
P_D = x^4 - 149 x^2 + 4900 = ( x^2 - 100 ) ( x^2 - 49 )
\]

and its spectrum is

\[
S_D = \{ \pm 10, \pm 7 \}.
\]

If \( n=2 \), then \( r_2 + s'_2 \sqrt{2} = ( 2+1 \sqrt{2} )(3 + 2 \sqrt{2} )^2 = 58 + 41 \sqrt{2} \). The ordered pair \((r,s') = (58,41)\) is the solution of the Pell's equation (14). Using Theorem 1 we have \( a = 1682, b = 3362 \).

We have found non-balanced integral tree \( T(1;1682|3362) \). The characteristic polynomial of its divisor is

\[
P_D = x^4 - 5045 x^2 + 5654884 = ( x^2 - 3364 ) ( x^2 - 1681 ) = ( x^2 - 58^2 ) ( x^2 - 41^2 )
\]

and its spectrum is

\[
S_D = \{ \pm 58, \pm 41 \}.
\]

If \( n=3 \), then \( r_3 + s'_3 \sqrt{2} = ( 10+7 \sqrt{2} )(3 + 2 \sqrt{2} )^3 = 338 + 239 \sqrt{2} \). The ordered pair \((r,s') = (338,239)\) is the solution of the Pell's equation (14). Using Theorem 1 we have \( a = 57222, b = 114242 \). We have found non-balanced integral tree \( T(1;57222|114242) \). The characteristic polynomial of its divisor is

\[
P_D = ( x^2 - 338^2 ) ( x^2 - 239^2 )
\]
and its spectrum is
\[ S_D = \{ \pm 338, \pm 239 \}. \]

Using computers we have found more trees that belong to this class (with the smallest number of vertices). Their list is in the table 1.

<table>
<thead>
<tr>
<th>Table 1.</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>50</td>
</tr>
<tr>
<td>1 682</td>
</tr>
<tr>
<td>57 122</td>
</tr>
<tr>
<td>1 940 450</td>
</tr>
<tr>
<td>65 918 162</td>
</tr>
</tbody>
</table>

**Corollary 2.** Let \( s_1 = 2 \). Using the same method as in the Corollary 1 the following results can be found. Their list is in the Table 2.

<table>
<thead>
<tr>
<th>Table 2.</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>486</td>
</tr>
<tr>
<td>4 726</td>
</tr>
<tr>
<td>4 656 966</td>
</tr>
<tr>
<td>4 563 350 046</td>
</tr>
<tr>
<td>4 471 177 446</td>
</tr>
</tbody>
</table>

**Corollary 3.** Let \( s_1 = 3 \). Using the same method as in the Corollary 1 the following results can be found. Their list is in the Table 3.

<table>
<thead>
<tr>
<th>Table 3.</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
</tr>
<tr>
<td>12</td>
</tr>
<tr>
<td>147</td>
</tr>
<tr>
<td>2 028</td>
</tr>
<tr>
<td>2 827</td>
</tr>
<tr>
<td>393 132</td>
</tr>
<tr>
<td>5 475 603</td>
</tr>
<tr>
<td>76 265 292</td>
</tr>
<tr>
<td>1 062 238 467</td>
</tr>
<tr>
<td>14 795 073 228</td>
</tr>
<tr>
<td>206 068 786 707</td>
</tr>
<tr>
<td>2 870 167 940 652</td>
</tr>
<tr>
<td>39 976 282 382 403</td>
</tr>
</tbody>
</table>

**Corollary 4.** Let \( s_1 = 4 \). Using the same method as in the Corollary 1 the following results can be found. Their list is in the Table 4.

Notice that the equation (11) has one of the following forms
\begin{align}
(15) & \quad 5 = r^2 - 5s^2. \\
(16) & \quad 5 = r^2 - 20s^2.
\end{align}

It is easy to show that the equation (16) can be construct from the equation (15) using substitution.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$a$ & $b$ & $r$ & $s'$ & $s$ \\
\hline
20 & 20 & 5 & 2 & 4 \\
5 780 & 7 220 & 85 & 38 & 76 \\
1 860 500 & 2 325 620 & 1 525 & 682 & 1 364 \\
599 074 580 & 748 843 220 & 27 365 & 12 238 & 24 476 \\
192 900 153 620 & 241 125 192 020 & 491 045 & 219 602 & 439 204 \\
62 113 250 390 420 & 77 641 562 988 020 & 8 814 445 & 3 940 598 & 7 881 196 \\
\hline
\end{tabular}
\caption{Table 4.}
\end{table}

Corollary 5. Let \( s_1 = 5 \). Using the same method as in the Corollary 1 the following results can be found. Their list is in the Table 5.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$a$ & $b$ & $r$ & $s'$ & $s$ \\
\hline
30 & 30 & 6 & 1 & 5 \\
13 230 & 15 870 & 126 & 23 & 115 \\
6 375 630 & 7 650 750 & 2 766 & 505 & 2 525 \\
3 073 039 230 & 3 687 647 070 & 60 726 & 11 087 & 55 435 \\
1 481 198 532 030 & 1 777 438 238 430 & 1 333 206 & 243 409 & 1 217 045 \\
713 934 619 398 030 & 856 721 543 277 630 & 29 269 806 & 5 343 911 & 26 719 555 \\
\hline
\end{tabular}
\caption{Table 5.}
\end{table}

Conclusion

In the present paper the class of integral trees of diameter 3 is studied. It is proved that the problem of characterizing integral trees of diameter 3 is equivalent with the problem of solving Pell’s Diophantine equations \( x^2 - D \cdot y^2 = L \) for appropriate integers \( D, L \). The paper also contains instructions for finding integral trees of diameter 3 and lists of some integral balanced trees with the “smallest” number of vertices.

References

INTEGRÁLNE STROMY A PELLOVE ROVNICE

Graf $G$ nazývame integrálny, ak všetky korene charakteristického polynomu $P(G; x)$ sú celočíselné. V tomto článku opisujeme triedu integrálnych stromov priemeru 3 a dokazujeme, že problém charakterizácie integrálnych stromov priemeru 3 je ekvivalentný s problémom riešenia Pellových diofantických rovníc. Článok zároveň obsahuje inštrukcie pre hľadanie integrálnych stromov priemeru 3 a zoznamy niekoľkých integrálnych stromov s “najmenším” počtom vrcholov.