REMARKS ON THE MEASURE DENSITY AND THE MAPPINGS ON THE SET OF POSITIVE INTEGERS

Milan Paštéka
Pedagogická fakulta Trnavskéj Univerzity
Priemyselná 4, P.O. BOX 4, Sk-914 43, Trnava,SR
e-mail: pasteka@mat.savba.sk

SUMMARY. In the first part we study the mappings which preserve zero asymptotic density and we give a characterization of the sets of zero asymptotic density in the terms of bijections. The object of observations in the second and third part is the uniform density

Let \( \mathbb{N} \) be the set of natural numbers. For any subset \( A \subseteq \mathbb{N} \) and \( x > 0 \), let \( A(x) \) be the cardinality of \( A \cap [0,x) \). The value \( \lim \sup_x x^{-1} A(x) := \overline{d}(A) \) is called the upper asymptotic density of \( A \), the value \( \lim \inf_x x^{-1} A(x) := \underline{d}(A) \) is called the lower asymptotic density of \( A \). If \( \overline{d}(A) = \underline{d}(A) \) then we say that \( A \) has an asymptotic density and the value \( \overline{d}(A) = \underline{d}(A) \) is called the asymptotic density of the set \( A \). It is easy to see that this if and only if the limit \( \lim \frac{A(x)}{x} := d(A) (= \overline{d}(A) = \underline{d}(A)) \) exists. For more details on the asymptotic density we refer to the paper [G].

Lemma 1. Suppose that \( \mathcal{A} \subseteq \mathbb{N} \) is an infinite and \( f : \mathcal{A} \to \mathbb{N} \) is such a mapping that

\[
\liminf_{\mathcal{A}} f(n) > 0.
\]

Then for every \( S \subseteq \mathcal{A} \) it holds

\[
d(S) = 0 \Rightarrow d(f(S)) = 0.
\]

Proof: The inequality (1) implies that for some \( \alpha > 0 \) we have \( n \cdot \alpha < f(n), n \in \mathcal{A} \). This implies that for \( x > 0 \) we have \( f(n) \leq x \) yields \( n \cdot \alpha < x \). Thus for \( S \subseteq \mathcal{A} \) we get \( f(S)(x) \leq S(\frac{x}{\alpha}) \). From this we immediately obtain (2). \( \square \)

If \( f : \mathbb{N} \to \mathbb{N} \) fulfills the condition (2) for every set \( S \subseteq \mathbb{N} \) we say that \( f \) preserves the zero density.

For every set \( S \subseteq \mathbb{N} \) it holds that \( d(S) = 0 \) if and only if \( d(\mathbb{N} \setminus S) = 1 \). From this we obtain immediately:

Lemma 2. Let \( f : \mathbb{N} \to \mathbb{N} \) be a permutation. Then \( f \) preserves the zero density if and only if for every \( R \subseteq \mathbb{N} \) it holds

\[
d(R) = 1 \Rightarrow d(f(R)) = 1.
\]
Theorem 1. Let \( g : \mathbb{N} \to \mathbb{N} \) be such a permutation that there exists a set \( \mathcal{A} \subset \mathbb{N} \), 
\( d(\mathcal{A}) = 1 = d(g(\mathcal{A})) \), and for every infinite \( S \subset \mathcal{A} \), 
\( d(S) = 0 \), we have

\[
(4) \quad \liminf_{S} \frac{g(n)}{n} > 0.
\]

Then \( g \) preserves the zero density.

Proof. Let \( R \subset \mathbb{N} \) and \( d(R) = 1 \). Then \( d(R \cap \mathcal{A}) = 1 \). Thus \( d(\mathbb{N} \setminus R \cap \mathcal{A}) = 0 \). From (4) and Lemma 1 we get \( d(g(\mathbb{N} \setminus R \cap \mathcal{A})) = 0 \). This yields \( d(g(R \cap \mathcal{A})) = 1 \), thus \( d(g(R)) = 1 \). The assertion follows from Lemma 2. \( \square \)

Example. Let \( \mathbb{N} \setminus \{n^2, n \in \mathbb{N}\} = A \cup B \) and \( \mathbb{N} \setminus \{n^3, n \in \mathbb{N}\} = C \cup D \), where \( A = \{a_1 < a_2 < \ldots \} \), \( B = \{b_1 < b_2 < \ldots \} \), \( C = \{c_1 < c_2 < \ldots \} \), \( D = \{d_1 < d_2 < \ldots \} \). 
Moreover \( A \cap B = 0 = C \cap D \). Let us consider the permutation \( g : \mathbb{N} \to \mathbb{N} \) where 
\( g(n^2) = n^3, n \in \mathbb{N} \) and \( g(a_k) = c_k, g(b_k) = d_k \). If we suppose that the sets \( A, B, C, D \)
have positive asymptotic density, then \( g \) fulfills the assumption of Theorem 1. If \( d(A) \neq d(C) \) then \( g \) preserves the zero density but does not preserve the asymptotic density.

Theorem 2. Let \( g : \mathbb{N} \to \mathbb{N} \) be an injective mapping and \( \mathcal{A} \subset \mathbb{N} \), \( \mathcal{A} = \{a_1 < a_2 < \ldots \} \) an infinite set.

a) If

\[
(5) \quad \lim_{n \to \infty} \frac{1}{a_n} \max\{g(a_j), j = 1, \ldots, n\} = 0
\]

then \( d(A) = 0 \).

b) If

\[
(6) \quad \lim_{n \to \infty} \frac{g(a_n)}{a_n} = 0
\]

then \( d(A) = 0 \).

Proof. a) The values \( g(a_j), j = 1, \ldots, n \) are different positive integers and so their maximum must be greater than \( n - 1 \). This implies

\[
\frac{n}{a_n} \leq \frac{1}{a_n} \max\{g(a_j), j = 1, \ldots, n\}.
\]

Now (5) implies \( d(A) = 0 \).

b) Put \( a_{k_n} \) such that \( g(a_{k_n}) = \max\{g(a_j), j = 1, \ldots, n\}, n = 1, 2, \ldots \) Then

\[
\frac{g(a_{k_n})}{a_n} \leq \frac{g(a_{k_n})}{a_{k_n}}
\]

because \( a_{k_n} \leq a_n \). The set \( \{g(a_{n}), n = 1, 2, \ldots \} \) infinite and so \( k_n \to \infty \) as \( n \to \infty \).
Therefore (6) implies (5). \( \square \)

As a corollary of Theorem 2 we obtain the following characterization of the sets of zero density in the terms of permutations.

Corollary. Let \( \mathcal{A} \subset \mathbb{N} \), \( \mathcal{A} = \{a_1 < a_2 < \ldots \} \) be an infinite set. Then \( d(A) = 0 \) if and only if there exists a permutation \( g : \mathbb{N} \to \mathbb{N} \) fulfilling (6).

Proof. The sufficiency follows from Theorem 2. If \( d(A) = 0 \) then \( a_{k_n} \to 0 \) for \( n \to \infty \). Put \( B = \mathbb{N} \setminus \mathcal{A} = \{b_n, n = 1, 2, \ldots \} \). The permutation \( g \) given by 
\( g(a_n) = 2n, g(b_n) = 2n + 1 \), fulfills (6). \( \square \)
Uniform density

Let $x < y$ be two positive real number, put $A(x, y) := A(y) - A(x)$, thus this value gives us the number of elements of $A$ between $x, y$.

Denote $\alpha_k(A) = \max_k A(k, k + s), \alpha_s(A) = \min_k A(k, k + s)$. It is well known that there exist the limits $\lim_{s \to 1} \frac{1}{s} \alpha_s(A) := \overline{\nu}(A)$ and $\lim_{s \to 1} \frac{1}{s} \alpha_s(A) := \underline{\nu}(A)$. The value $\overline{\nu}(A)$ is called the upper uniform density of $A$ and the value $\underline{\nu}(A)$ is called the lower uniform density of $A$. The definition implies:

i) If $A \subset \mathbb{N}$ and the set $A$ contains the blocks of consecutive numbers of arbitrary length then $\overline{\nu}(A) = 1$.

Let us denote $\nu(B) = \sum_{n \in B} \frac{1}{n}$, where the union is considered through all prime numbers.

ii) If $A \subset \mathbb{N}$ and the set $\mathbb{N} \setminus A$ contains the blocks of consecutive numbers of arbitrary length then $\underline{\nu}(A) = 0$.

Theorem 1. Let $A, B$ be two infinite subsets of $\mathbb{N}$ such that $A$ contains the blocks of consecutive elements from $B$ of arbitrary length. Then $\overline{\nu}(A) \geq \nu(B)$.

Proof: The assumptions yield that for arbitrary $n$ it is such $k$ that $A(k, k + n) \geq B(k, k + n)$, thus max $A(k, k + n) \geq \min B(k, k + n)$ and the assertion follows. □

If for $A \subset \mathbb{N}$ it holds $\underline{\nu}(A) = \nu(A) := u(A)$ then we say that $A$ has uniform density, and the value $u(A)$ is called the uniform density of $A$.

Let $A = \{a_1 < a_2 < \ldots\}$ be an infinite set. It is well known fact that if $\sum_{n} a_n^{-1} < \infty$ then $A$ has the asymptotic density and $d(A) = 0$. Now we give an example that this does not hold for the uniform density. Consider the set $A = \{n! + 1, \ldots, n! + n\}$. From i) we see that $\overline{\nu}(A) = 1$ but it is easy to prove that in this case $\sum_{n} a_n^{-1} < \infty$.

Theorem 2. Let $\{m_n\}$ be a sequence of positive integers, such that $(m_j, m_k) = 1$ for $k \neq j$. Put $A = \cup_{n=1}^\infty m_n \mathbb{N}$. Then

1. $\overline{\nu}(A) = 1$
2. $\underline{\nu}(A) = 1 - \prod_{n=1}^\infty (1 - \frac{1}{m_n})$.

Proof: (1). The numbers $m_1, \ldots, m_n$ are relatively prime, thus due to the Chinese reminder theorem we obtain that there exists such a positive integer $x_n$ that $x_n \equiv -j \pmod{m_j}$ for $j = 1, \ldots, n$. Therefore $x_n + j \in m_j \mathbb{N}$, $j = 1, \ldots, n$. This yields $x_n + 1, \ldots, x_n + n \in A$ and from i) we obtain $\overline{\nu}(A) = 1$.

(2). Put $A_n = \cup_{j=1}^n m_j \mathbb{N}$. Clearly $A_n \subset A$. It can be easily proved $u(A_n) = 1 - \prod_{j=1}^n (1 - \frac{1}{m_j})$ and so for $n \to \infty$ we obtain $1 - \prod_{n=1}^\infty (1 - \frac{1}{m_n}) \leq \nu(A)$. Other inequality we obtain from the fact that $d(A) = 1 - \prod_{n=1}^\infty (1 - \frac{1}{m_n})$. □

Denote by $Q_n$, for $n = 2, 3, \ldots$ the set of positive integers which are not divisible by the $n$-th power of prime number. Denote by $\mathcal{P}$ the set of all prime numbers. Then it holds $N \setminus Q_n = \cup_{p \in \mathcal{P}^{n}} p^n \mathbb{N}$, where the union is considered through all prime numbers $p$. Thus $\overline{\nu}(N \setminus Q_n) = 1 - \prod_{p \in \mathcal{P}} (1 - p^{-n}) > 0$ and so from ii) it follows that $Q_n$ does not contains the blocks of consecutive integers of arbitrary length.

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Now we shall study one type of arithmetic functions from point of view of the uniform density of their range.

Lemma 1. Let $f : \mathbb{N} \to \mathbb{N}$ be an arithmetic function fulfilling the condition

(a) $\liminf_{n \to \infty} \frac{f(n+k) - f(k)}{n} > 0$ uniformly for $k = 1, 2, \ldots$. 

Then for every \( A \subseteq \mathbb{N} \), \( u(A) = 0 \) it holds \( u(f(A)) = 0 \).

**Proof:** The condition (a) implies that for suitable \( \beta > 0, n_0 \in \mathbb{N} \) we have
\[
(1) \quad f(n + k) - f(k) \geq \beta n, \quad n \geq n_0, \quad k = 1, 2, \ldots
\]
Thus the set \( F := f(\mathbb{N}) \) can be represented in the form \( F = F^{(1)} \cup \cdots \cup F^{(n_0)} \) where
\[
F^{(i)} = \{ f(i) < f(i + n_0) < \cdots < f(i + m n_0) < \cdots \},
\]
for \( i = 1, \ldots, n_0 \). Let us denote \( E^{(i)} = F^{(i)} \cap f(A) \). Thus \( E^{(i)} = \{ f(i + m n_0); i + m n_0 \in A, m \in \mathbb{N} \}, i = 1, \ldots, n_0 \). Clearly \( f(A) \subseteq E^{(1)} \cup \cdots \cup E^{(n_0)} \), therefore it suffices to prove \( u(E^{(i)}) = 0 \), \( i = 1, \ldots, n_0 \).

Let \( k, n \in \mathbb{N} \) and
\[
f(i + m_1 n_0), \ldots, f(i + m_s n_0) \in [k, k + n]
\]
for \( m_1 < m_2 < \ldots m_s, m_j \in \mathbb{N}, i + m_j n_0 \in A, j = 1, \ldots, s \). Then
\[
f(i + m_s n_0) - f(i + m_1 n_0) \leq n.
\]
From the other side the inequality (1) implies
\[
f(i + m_s n_0) - f(i + m_1 n_0) \geq \beta(m_s - m_1)n_0.
\]
This yields \( \beta(m_s - m_1)n_0 \leq n \) and so \( m_s \leq m_1 + \frac{n}{\beta n_0} \). The numbers \( i + m_j n_0, j = 1, \ldots, s \) belong to the interval \( [r, r + \frac{n}{\beta}] \), where \( r = i + m_1 n_0 \). We get \( s \leq A(r, r + n) \), in the other words
\[
E^{(i)}(k, k + n) \leq A(r, r + \frac{n}{\beta}),
\]
thus \( u(E^{(i)}) = 0 \). \( \Box \)

Now we recall a well known property of uniform density. Denote for a prime number \( p \) and \( A \subseteq \mathbb{N} \) by \( A_p \) the set of these elements of \( A \) which are divisible by \( p \) and not divisible by \( p^2 \).

In [P] it is proved the following statement: Let \( P \) be such set of primes that \( \sum p^{-1} = \infty \). Then for \( A \subseteq \mathbb{N} \) it holds
\[
(3) \quad (\forall p \in P; u(A_p) = 0) \Rightarrow u(A) = 0.
\]

**Lemma 2.** Let \( P \) be such set of primes that \( \sum p^{-1} = \infty \). Denote for \( r = 1, 2, \ldots \) by \( N(r) \) the set of all positive integers which have at most \( r \) distinct prime divisors from \( P \). Then \( u(N(r)) = 0, r = 1, 2, \ldots \).

**Proof:** By induction with respect to \( r \). Clearly \( u(N(0)) = 0 \), for \( p \in P \), thus (3) yields \( u(N(0)) = 0 \).

It is easy to see that \( N(r + 1)_p \subseteq p N(r) \), thus from (3) we obtain \( u(N(r)) = 0 \) \( \Rightarrow \)
\( u(N(r + 1)) = 0, r = 1, 2, \ldots \). \( \Box \)

**Theorem 3.** Let \( f : \mathbb{N} \rightarrow \mathbb{N} \) be an arithmetic function fulfilling the condition (a) from Lemma 1. Let \( P \) be such set of primes that \( \sum p^{-1} = \infty \). Denote by \( \omega(n) \) the number of distinct prime divisors from \( P \) of \( n, n \in \mathbb{N} \). Let \( f \) fulfills moreover the condition
\[
(b) \quad \text{There exists } a \in \mathbb{N}, a > 1 \text{ such that } a^g(\omega(n))|f(n) \text{ for } n \in \mathbb{N}. \quad \text{Where } g : \mathbb{N} \rightarrow \mathbb{N} \text{ is a function that } g(n) \rightarrow \infty \text{ for } n \rightarrow \infty.
\]

Then \( u(F) = 0 \), where \( F = \{ f(n), n \in \mathbb{N} \} \).

**Proof:** Let \( s \in \mathbb{N} \). The set \( F \) can be decomposed to \( F = F_1 \cup F_2 \), where \( F_1 = \{ f(j); j \in \mathbb{N}, \omega^s(f(j)) \} \) and \( F_2 = F \setminus F_1 \). Clearly \( \overline{\mathbb{N}}(F_1) \leq a^{-s} \). We prove \( u(F_2) = 0 \). The condition (b) yields that there exists a nonnegative integer \( r \) that \( F_2 \subseteq f(\mathbb{N}(r)) \), where \( \mathbb{N}(r) \) is the set from Lemma 2. Thus Lemma 1 implies \( u(F_2) = 0 \). Therefore \( \overline{\mathbb{N}}(F) \leq a^{-s} \) and for \( s \rightarrow \infty \) we obtain \( u(F) = 0 \). \( \Box \)
**Transformations which preserve the uniform density**

We conclude this note by one sufficient condition under which an injective mapping preserves the uniform density.

**Theorem 1.** Let \( g : \mathbb{N} \to \mathbb{N} \) be an injection fulfilling the condition

\[
\lim_{n \to \infty} \frac{g(n + k) - g(k)}{n} = 1
\]

uniformly for \( k = 1, 2, \ldots \). Then \( g \) preserves the uniform density.

For the proof we shall use the following statement proved in the paper [GLS].

**Lemma.** Let \( S = \{ s_1 < s_2 < \ldots \} \subset \mathbb{N} \) be an infinite set. The \( S \) has the uniform density if and only if the fraction

\[
\frac{n}{s_{n+k} - s_k}
\]

converges uniformly as \( n \to \infty \), \( k = 1, 2, \ldots \). And in this case the value of its limit is equal to the uniform density of \( S \).

**Proof of Theorem 1.** The condition (1) yields that for two sequences \( \{ h_1(n, k) \}, \{ h_2(n, k) \} \) such that \( h_1(n, k) - h_2(n, k) \to \infty, n \to \infty \) uniformly for \( k = 1, 2, \ldots \) we have

\[
\frac{g(h_1(n, k)) - g(h_2(n, k))}{h_1(n, k) - h_2(n, k)} \equiv 1, \quad n \to \infty
\]

(As usually we use the symbol \( \equiv \) for the uniform convergence.)

Let \( A = \{ a(1) < a(2) < \ldots \} \) be an infinite set, which has the uniform density and \( u(A) = \alpha \).

From Lemma we obtain

\[
\frac{n}{a(n + k) - a(k)} \equiv \alpha, \quad n \to \infty
\]

Put \( g(A) = \{ g(a(1)), g(a(2)), \ldots \} \). These elements are not necessarily arranged to their magnitude. Clearly \( a(n + k) - a(k) \geq n \), and so \( a(n + k) - a(k) \equiv \infty \) as \( n \to \infty \). The relation (2) now implies

\[
\frac{g(a(n + k)) - g(a(k))}{a(n + k) - a(k)} \equiv 1, \quad n \to \infty
\]

Therefore for suitable \( n_0 \) the fraction on left side is positive for \( k = 1, 2, \ldots, \) thus \( g(a(n_0 + k)) > g(a(k)), k = 1, 2, \ldots \). And so we see that the set \( g(A) \) we can decompose into a union of disjoint sets

\[
g(A) = B_1 \cup B_2 \cup \cdots \cup B_{n_0}
\]

where

\[
B_j = \{ g(a(j)) < g(a(j + n_0)) < \cdots g(a(j + rn_0)) \cdots \} \quad j = 1, \ldots, n_0.
\]
The relation (3) now implies

\[ \frac{r \cdot n_0}{a(j + (r + k)n_0) - a(j + k \cdot n_0)} \to \alpha, \quad r \to \infty \]

Moreover the relation (2) yields

\[ \frac{g(a(j + (k + r)n_0) - g(a(j + k \cdot n_0))}{a(j + (k + r)n_0) - a(j + k \cdot n_0)} \to 1, \quad r \to \infty \]

because the denominator is \( \geq r \cdot n_0 \) and so tends to \( \infty \) uniformly for \( k = 1, 2, \cdots \).

Thus from (6) and (7) we can deduce

\[ \frac{r}{g(a(j + (k + r)n_0) - g(a(j + k \cdot n_0))} \to \frac{\alpha}{n_0}, \quad r \to \infty \]

and so \( u(B_j) = \frac{\alpha}{n_0}, j = 1, \cdots , n_0 \). From (5) we have \( u(g(A)) = \alpha \). \( \Box \)

Consider \( g(n) = n + c \cdot \log n + O(1) \). Then \( g(n + k) - g(k) = n + c \cdot \log (\frac{n}{k} + 1) + O(1) \), but \( O \leq \log (\frac{n}{k} + 1) \leq \log (n + 1) \) and \( g \) fulfills (1). Analogously it can be proved that

\[ g(n) = n + c_1 \log_{r_1} n + c_2 \log_{r_2} (n) + \cdots + O(1) \]

where \( r_1, r_2, \cdots , r_j > 1 \) fulfills (1).

References


