ABOUT AN EXPERIMENT WITH THE HARMONIC SERIES

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Abstract: This article describes a connection between the harmonic series and the Euler’s number e.
Key words: Sequence, harmonic series, Euler’s number

1. Introduction

In this contribution we are to describe a case when a computer was used for discovering a hypothesis in secondary school mathematics lessons. This hypothesis refers to the harmonic series

\[ 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots \]

which is a well-known example of a divergent series (with sum \(+\infty\)) although it meets the necessary condition of convergence, i.e. a sequence of its \(n^{th}\) terms converges to zero. As you know, a sequence of partial sums \(s_n\) grows up very slowly.

\[ s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \]

In mathematical analysis textbooks (see e.g. [1]) we can learn that

\[ s_{1000} = 7.48 \ldots, \quad s_{1000000} = 14.39 \ldots. \]

In secondary school mathematics lessons we tried to demonstrate the growth of partial sums \(s_n\) with the help of computer technology.

2. Description of the experiment

If we wish to obtain values of \(s_n\), we can use a calculator for first computations. Like that, we can elicit for instance (correct to seven decimal places):

\[
\begin{align*}
s_1 &= 1.0000000 & s_6 &= 2.4500000 & s_{11} &= 3.0198773 \\
s_2 &= 1.5000000 & s_7 &= 2.5928571 & s_{12} &= 3.1032107 \\
s_3 &= 1.8333333 & s_8 &= 2.7178571 & s_{13} &= 3.1801338 \\
s_4 &= 2.0833333 & s_9 &= 2.8289683 & s_{14} &= 3.2515623 \\
s_5 &= 2.2833333 & s_{10} &= 2.9289683 & &
\end{align*}
\]
We were concerned by sums $s_1, s_4, s_{11}, \ldots$, where the partial sum reaches values of 1, 2, 3, \ldots for the first time. The relevant indexes 1, 4, 11, \ldots were marked as $p_1, p_2, p_3, \ldots$. So $p_n$ is the index of such a partial sum of the harmonic series, for which the following is true:

$$s_{p_n - 1} < n \leq s_{p_n}.$$ 

For getting other values of terms of the sequence $p_n \propto n^{-1}$ the calculator is no longer sufficient because the sequence $p_n \propto n^{-1}$ grows up very quickly, i.e. to get values of other terms it is necessary to add more and more terms of the harmonic series.

That’s why we have set a programme (it is a GW Basic version for computation of $M$ terms)

10 REM Harmonic series
20 CLS : PRINT "Harmonic series" : PRINT
30 S#=0 : P#=1 : N=1 : M=12
40 S#=S#+1/P# : P#=P#+1 : IF S#<N THEN 40
50 PRINT N,P#
60 N=N+1 : IF N<=M THEN 40
70 END

Calculations of numbers $p_n$ with a growing $n$ escalate their time demand; the time consumption is dependent also on the kind of a computer that is being used though. In the columns you can see the first 18 terms of the sequence $p_n \propto n^{-1}$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$p_n$</th>
<th>$p_n \propto n^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>227</td>
<td>33,617</td>
</tr>
<tr>
<td>4</td>
<td>616</td>
<td>91,380</td>
</tr>
<tr>
<td>11</td>
<td>1674</td>
<td>248,397</td>
</tr>
<tr>
<td>31</td>
<td>4550</td>
<td>675,214</td>
</tr>
<tr>
<td>83</td>
<td>12367</td>
<td>835,421</td>
</tr>
</tbody>
</table>

We have noticed that every term of this sequence is approximately triple to the term preceding. For this hypothesis verification we carried out calculations of $\frac{p_{n+1}}{p_n}$ (applying computer technology again) and we got the sequence (in columns again):

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\frac{p_{n+1}}{p_n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2.7136...</td>
</tr>
<tr>
<td>2.75</td>
<td>2.7175...</td>
</tr>
<tr>
<td>2.8181...</td>
<td>2.7182675...</td>
</tr>
<tr>
<td>2.6774...</td>
<td>2.7182825...</td>
</tr>
<tr>
<td>2.7349...</td>
<td>2.7182825...</td>
</tr>
</tbody>
</table>

At first sight it seems that its terms approach the Euler’s number $e$. Therefore it was possible to state the hypothesis
\[
\lim_{n \to \infty} \frac{p_{n+1}}{p_n} = e. \tag{1}
\]

Looking at it more closely though, this result is not so surprising, as we realize the connection between partial sums of the harmonic series and the integral

\[
\int_1^A \frac{dx}{x} = \ln A.
\]

If the hypothesis (1) is correct, it means that \( p_n \) is a certain “quasi-geometric” sequence with the common ratio \( e \). It is possible to assess the value of its other terms approximately, without any more computation of harmonic series partial sums.

It is really possible to show (see [2]) that

\[
1 - \frac{1}{p_n} < \ln \frac{p_{n+1}}{p_n} < 1 + \frac{1}{p_n} \tag{2}
\]

holds for every natural number \( n \).

Because the sequence \( \left\{ \frac{1}{p_n} \right\}_{n=1}^{\infty} \) converges to zero, out of the three-sequence theorem it follows that

\[
\lim_{n \to \infty} \ln \frac{p_{n+1}}{p_n} = 1
\]

and the hypothesis (1) is therefore proved.

Now let’s have a look at how to designate the other terms of the sequence \( p_n \). Out of (2), through various conversions, we get

\[
p_n e^{1/p_n} < p_{n+1} < p_n e^{1/p_n}, \tag{3}
\]

from which we can assess the term \( p_{n+1} \) once we know the term \( p_n \).

Let’s test this assessment at computing \( p_{10} \) by that of \( p_9 \). According to (3),

\[
4550 e^{1/4550} < p_{10} < 4550 e^{1/4550}
\]

and so \( p_{10} \in \{12366, 12367, 12368, 12369, 12370\} \). We could see that the right value is \( p_{10} = 12367 \).

Let’s consider the question of the \( p_{n+1} \) designation accuracy by applying the formula (3). We are interested in the \( 2 \delta_n = \beta_n - \alpha_n \) length of the interval \((\alpha_n, \beta_n)\) where

\[
\alpha_n = p_n e^{1/p_n}
\]
Applying the l’Hospital’s rule we come to

$$\lim_{n \to \infty} 2\delta_n = 2e.$$ 

Consequently, if we set $p_{n+1} \approx \frac{1}{2} \ln + \beta_n$, we depart from the real value by less than $\delta_n$ where $\delta_n$ converges to $e$.

### 3. Conclusion

In the process of the experiment we discovered with the students a nice connection between the harmonic series and the Euler’s number $e$. This important constant was discovered in the process of the experiment in the natural way. This article, therefore, can be an appeal for discovering similar hypothesis of number series with the help of computer technology.

### References